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Shape Invariant and Rodrigues Solution of the Dirac-shifted Oscillator and Dirac-Morse Potentials

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Abstract We show that the Dirac equation for a charged spinor in spherically symmetric electromagnetic potentials as Dirac-shifted oscillator and Dirac-Morse potentials have the conditions of shape invariant symmetry in non-relativistic quantum mechanics. The relativistic spectra of the bound states and spinor wavefunctions can be obtained by the Rodrigues polynomials of one associated differential equation.

Keywords Dirac-shifted oscillator · Dirac-Morse · Rodrigues representation · Shape invariance

1 Introduction

Recently the ideas of supersymmetry and shape invariance in non-relativistic quantum mechanics [3] have developed considerably and have been expanded to other branches of physics. By using this algebraic method, a large number of non-relativistic shape invariant potentials have been solved exactly and their energy spectra and wavefunctions have been obtained [4, 5]. Relativistic extensions of these potentials are very useful to study the relativistic effects, therefore physicists have been tried to solve Dirac equation for some physical potentials such as: Harmonic oscillator [10], Coulomb [11], Morse [1], four-parameter diatomic potential [6], symmetrical double well potential [12], etc. Also, the Dirac equation for a charged spinor in spherically symmetric electromagnetic potential was solved by Alhaidari [2], and the relativistic spectra of the bound states and spinor wavefunctions for some potential such as: Dirac–Rosen-Morse, Dirac–Eckart, Dirac–Scarf and Dirac–Poschl–Teller have been obtained. On the other hand in reference [8], the authors have shown that the standard second order differential equations of mathematical physics and their associated differential equations have the properties of supersymmetry and shape invariance symmetry.

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By using these properties, the authors have shown that the associated differential equations can be factorized as the operator product of raising and lowering operators and a list of known one dimensional shape invariant potentials are given in Ref. [9].

In this work, we solve the Dirac equation for two shape invariant potentials by letting two different choices for electrostatic potentials and relativistic energies. We try to investigate their shape invariance properties by factorization method and obtain their spinor wavefunctions in terms of Rodrigues representation.

The paper is organized as follows: in Sect. 2, we present a unitary transformation on Dirac equation with spherically symmetric potentials and obtain a Schrodinger-like differential equation for the upper spinor component. In Sect. 3, we try to obtain the spinor wavefunctions of Dirac-shifted oscillator and Dirac-Morse potentials by using of the shape invariance symmetry and the Rodrigues polynomials and finally the paper ends with a brief conclusion in Sect. 4.

2 Dirac Equation with Spherically Symmetric Potentials

At the first, let us follow the procedure of solving the Dirac equation for shape invariant potentials and obtain a Schrodinger-like second order differential equation for the upper spinor component that is given by Alhaidari [2]. Supposing that $m = e = \hbar = 1$ and $c = \alpha^{-1}$, the Hamiltonian for a Dirac-particle in the four-component electromagnetic potential $(A_0, \vec{A})$ can be written as:

$$H = \begin{pmatrix} 1 + \alpha A_0 & -i\alpha\vec{\sigma}\cdot\vec{\nabla} + \alpha\vec{\sigma}\cdot\vec{A} \\ -i\alpha\vec{\sigma}\cdot\vec{\nabla} + \alpha\vec{\sigma}\cdot\vec{A} & -1 + \alpha A_0 \end{pmatrix}, \tag{1}$$

where α is fine structure constant and $\vec{\sigma}$ are the three 2×2 Pauli matrices. Taking into consideration gauge invariance, the form of electromagnetic potential for static charge distribution with spherical symmetry is:

$$(A_0, \vec{A}) = (\alpha V(r), \hat{r} W(r)), \tag{2}$$

where $\hat{r}$ is radial unit vector, $V(r)$ is electrostatic potential and $W(r)$ is a gauge field. By substituting the two off-diagonal term $\alpha\vec{\sigma}\cdot\vec{A}$ by $\pm i\alpha\vec{\sigma}\cdot\vec{A}$ in (1), the Hamiltonian leads to the following two component radial Dirac equation [7]:

$$\begin{pmatrix} 1 + \alpha^2 V(r) & \alpha[\frac{k}{r} + W(r) - \frac{d}{dr}] \\ \alpha[\frac{k}{r} + W(r) + \frac{d}{dr}] & -1 + \alpha^2 V(r) \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = \varepsilon \begin{pmatrix} g(r) \\ f(r) \end{pmatrix}, \tag{3}$$

where ε is relativistic energy and k is spin-orbit coupling parameter defined as: $k = \pm(j + 1/2)$ for $l = j \pm 1/2$. Equation (3) gives two coupled first order differential equations for the two radial spinor components. By eliminating the lower component, one can obtain a second order differential equation for the upper. The first order derivative can be eliminated by a general local unitary transformation as:

$$\begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = \begin{pmatrix} \cos(\rho(x)) & \sin(\rho(x)) \\ -\sin(\rho(x)) & \cos(\rho(x)) \end{pmatrix} \begin{pmatrix} \phi(x) \\ \theta(x) \end{pmatrix}, \quad r = q(x). \tag{4}$$

The stated requirement gives the following constraint:

$$\frac{dq}{dx}\left[-\alpha^2 V + \cos(2\rho) + \alpha \sin(2\rho)\left(W + \frac{k}{q}\right) + \alpha\frac{d\rho/dx}{dq/dx} + \varepsilon\right] = \text{constant} \equiv \eta \neq 0. \tag{5}$$

This transformation and resulting constraint are the relativistic analog of a point canonical transformation in non-relativistic quantum mechanics. Considering the case of a global unitary transformation defined by: $q(x) = x$ and $d\rho/dx = 0$ and substituting into (5) yields:

$$W(r) = \frac{\alpha}{S} V(r) - \frac{k}{r}, \tag{6}$$

$$\eta = C + \varepsilon, \tag{7}$$

where $S \equiv \sin(2\rho)$ and $C = \cos(2\rho)$. By applying above transformation and constraint, the radial Dirac equation (3) becomes:

$$\begin{pmatrix} C + 2\alpha^2 V & \alpha[-\frac{S}{\alpha} + \frac{\alpha C}{S} V - \frac{d}{dr}] \\ \alpha[-\frac{S}{\alpha} + \frac{\alpha C}{S} V + \frac{d}{dr}] & -C \end{pmatrix} \begin{pmatrix} \phi(r) \\ \theta(r) \end{pmatrix} = \varepsilon \begin{pmatrix} \phi(r) \\ \theta(r) \end{pmatrix}. \tag{8}$$

Equation (8) gives the lower spinor component in terms of the upper as follows:

$$\theta(r) = \frac{\alpha}{C + \varepsilon} \left[-\frac{S}{\alpha} + \frac{\alpha C}{S} V + \frac{d}{dr} \right] \phi(r), \tag{9}$$

while the equation for the upper component can be written as:

$$\left[-\frac{d^2}{dr^2} + \frac{\alpha^2}{T^2} V^2 + 2\varepsilon V - \frac{\alpha}{T} \frac{dV}{dr} - \frac{\varepsilon^2 - 1}{\alpha^2} \right] \phi(r) = 0, \tag{10}$$

where $T \equiv \frac{S}{C} = \tan(2\rho)$.

Equation (10) is a Schrodinger-like second order differential equation for the upper spinor component with potential:

$$\frac{\alpha^2}{T^2} V^2 + 2\varepsilon V - \frac{\alpha}{T} \frac{dV}{dr}$$

and energy spectrum:

$$\frac{\varepsilon^2 - 1}{\alpha^2}.$$

In the next section, we want to solve this equation for two given electrostatic potentials and relativistic energies.

3 Shape Invariant Symmetry and Rodrigues Solution: Two Different Dirac Potentials

In this section, we introduce two different kinds of Dirac-electrostatic potentials and try to obtain the spinor wave functions by shape invariant symmetry and Rodrigues polynomials as follow to Ref. [9].

3.1 Dirac-Shifted Oscillator Potential

Let us consider the electrostatic potential and relativistic energy in (10) as:

$$V(r) = -\frac{T}{\alpha} \left[\frac{1}{2} \gamma r - \lambda \right] - \frac{\varepsilon T^2}{\alpha^2}, \tag{11}$$

$$\varepsilon^2 = \frac{\alpha^2}{(T^2 + 1)} \left[\gamma(n - m + 1) + \frac{1}{\alpha^2} \right], \tag{12}$$

where $\gamma > 0$ and λ are constant parameters and supposing that n and m are integer numbers so that ε^2 be positive. Equation (10) can be written as the following simple form:

$$-\frac{d^2}{dr^2}\phi_{n.m}(r) + V_m(r)\phi_{n,m}(r) = E(n,m)\phi_{n,m}(r), \tag{13}$$

where:

$$V_m(r) - E(n,m) = \frac{\alpha^2}{T^2}V^2 + 2\varepsilon V - \frac{\alpha}{T}\frac{dV}{dr} - \frac{\varepsilon^2 - 1}{\alpha^2}. \tag{14}$$

In (14), $V_m(r)$ and $E(n,m)$ are potential and energy respectively, and supposing that $E(n,m)$ depends on n and m. By inserting the given formulas for electrostatic potential and relativistic energy according to (11), (12) into (14), we obtain the following differential equation from (13):

$$-\frac{d^2}{dr^2}\phi_{n,m}(r) + \left[\left(\frac{1}{2}\gamma r - \lambda\right)^2 + \frac{\gamma}{2}\right]\phi_{n,m}(r) = \gamma(n-m+1)\phi_{n,m}(r). \tag{15}$$

Equation (15) is the Schrodinger equation for shifted oscillator potential with energy spectrum of $\gamma(n-m+1)$, where we want to obtain its wavefunction $\phi_{n,m}(r)$ by Rodrigues representation.

If we define the first order differential operators A and $A^\dagger$ as:

$$A = -\frac{d}{dr} + \left(\frac{1}{2}\gamma r - \lambda\right), \tag{16}$$

$$A^\dagger = \frac{d}{dr} + \left(\frac{1}{2}\gamma r - \lambda\right), \tag{17}$$

then (15) can be factorized as:

$$A^\dagger A\phi_{n,m}(r) = \gamma(n-m+1)\phi_{n,m}(r) = E(n,m)\phi_{n,m}(r), \tag{18}$$

$$AA^\dagger\phi_{n,m-1}(r) = \gamma(n-m+1)\phi_{n,m-1}(r) = E(n,m)\phi_{n,m-1}(r). \tag{19}$$

These equations show the shape invariance conditions for shifted oscillator potential and it is quite obvious to see that for a given n the operator $A^\dagger$ raises the index m, while the operator A lowers it. Since $E(n,m)$ vanishes for $m = n+1$, therefore m takes values between zero and n so:

$$A^\dagger\phi_{n,n}(r) = 0, \tag{20}$$

where $\phi_{n,n}$ is the highest state and the upper spinor wavefunctions of Dirac-shifted oscillator can be obtained by operating the lowering operator on the highest state consecutively

$$\phi_{n,m}(r) = \frac{A}{E(n,m+1)}\cdots\frac{A}{E(n,n-1)}\frac{A}{E(n,n)}\phi_{n,n}(r). \tag{21}$$

In order to express the wavefunctions $\phi_{n,m}(r)$in terms of the known mathematical physics special functions, we define $\psi_{n,m}(x) = \exp(\frac{1}{4}\gamma x^2)\phi_{n,m}(r)$ and change the variable, $x = r - \frac{2\lambda}{\gamma}$ in (15). After some calculations, we obtain the associated Hermite differential equation as follows:

$$\frac{d^2}{dx^2}\psi_{n,m}(x) - \gamma x\frac{d}{dx}\psi_{n,m}(x) + \gamma(n-m)\psi_{n,m}(x) = 0. \tag{22}$$

The Rodrigues representation of the associated polynomials $\psi_{n,m}(x)$ is given by:

$$\psi_{n,m}(x) = (-1)^m \left(\frac{d}{dx}\right)^m \psi_n(x), \tag{23}$$

where $\psi_n(x)$ satisfies the Hermite differential equation which its Rodrigues representation is:

$$\psi_n(x) = N \exp\left(\frac{1}{2}\gamma x^2\right)\left(\frac{d}{dx}\right)^n \left(\exp\left(-\frac{1}{2}\gamma x^2\right)\right), \tag{24}$$

where N is a constant and γ is a positive parameter. It is obvious that the transformation $\psi_{n,m}(x) = \exp(\frac{1}{4}\gamma x^2)\phi_{n,m}(r)$, as a similarity transformation over (15), does not change the energy eigenvalues and only transforms raising and lowering operators. The upper spinor wavefunctions of Dirac-shifted oscillator problem can be expressed in terms of the Hermite orthogonal polynomials, which its Rodrigues representation is given by (23), (24). The lower spinor wavefunctions can also be obtained from (9).

3.2 Dirac-Morse Potential

In this case, let us consider the following choices for electrostatic potential and relativistic energy:

$$V(r) = -\frac{T}{\alpha}\left[-\lambda - \frac{\beta}{2}(\exp(-r))\right] - \frac{T^2}{\alpha^2}\varepsilon, \tag{25}$$

$$\varepsilon^2 = \frac{\alpha^2}{(T^2+1)}\left[-(\gamma+n+m)(n-m+1) + \frac{1}{\alpha^2}\right], \tag{26}$$

where $\gamma < -2$ and $\beta > 0$ are constant parameters, m and n are integer numbers so that ε^2 be positive also we suppose that $\lambda = m + \frac{\gamma}{2} - \frac{1}{2}$.

By substituting (25) and (26) into (14), (13) can be written as:

$$\begin{aligned} -\frac{d^2}{dr^2}\phi_{n,m}(r) + \left[\lambda^2 + \frac{\beta^2}{4}(\exp(-2r)) + \frac{\beta}{2}(1+2\lambda)(\exp(-r))\right]\phi_{n,m}(r) \\ = -(\gamma+n+m)(n-m+1)\phi_{n,m}(r), \end{aligned} \tag{27}$$

which is the Schrodinger equation for Morse potential. It can be factorized as the operator product of $A(m)$ and $A^\dagger(m)$:

$$A^\dagger(m)A(m)\phi_{n,m}(r) = -(\gamma+n+m)(n-m+1)\phi_{n,m}(r) = E(n,m)\phi_{n,m}(r), \tag{28}$$

$$A(m)A^\dagger(m)\phi_{n,m-1}(r) = -(\gamma+n+m)(n-m+1)\phi_{n,m-1}(r) = E(n,m)\phi_{n,m-1}(r), \tag{29}$$

where the raising and lowering operators $A^\dagger(m)$ and $A(m)$ are given by:

$$A(m) = -\frac{d}{dr} + \left(-\lambda - \frac{\beta}{2}(\exp(-r))\right), \tag{30}$$

$$A^\dagger(m) = \frac{d}{dr} + \left(-\lambda - \frac{\beta}{2}(\exp(-r))\right). \tag{31}$$

Equations (28), (29) show the shape invariance symmetry for Morse potential and energy eigenvalue $E(n,m)$ vanishes for $m=n+1$, that is:

$$A^{\dagger}(n+1)\phi_{n,n}(r)=0. \tag{32}$$

Therefore, similar to Dirac-shifted oscillator, solution of Dirac-Morse potential can be written as:

$$\phi_{n,m}(r)=\frac{A(m+1)}{E(n,m+1)}\cdots\frac{A(n-1)}{E(n,n-1)}\frac{A(n)}{E(n,n)}\phi_{n,n}(r). \tag{33}$$

Now, by introducing $\psi_{n,m}(x)=x^{-\frac{1}{2}(\gamma+1)}(\exp(\frac{\beta}{2x}))\phi_{n,m}(r)$ and letting $x=\exp(r)$ in (27), we obtain the following differential equation:

$$x^2\frac{d^2}{dx^2}\psi_{n,m}(x)+[(\gamma+2)x+\beta]\frac{d}{dx}\psi_{n,m}(x)-\left[\beta m\frac{1}{x}+n(\gamma+n+1)\right]\psi_{n,m}(x)=0. \tag{34}$$

According to the standard calculations, the Rodrigues representation of $\psi_{n,m}$ can be obtained as:

$$\psi_{n,m}(x)=(-1)^m x^m\left(\frac{d}{dx}\right)^m\psi_n(x), \tag{35}$$

where

$$\psi_n=N\left(x^{-\gamma}\exp\left(\frac{\beta}{x}\right)\right)\left(\frac{d}{dx}\right)^n\left(x^{\gamma+2n}\exp\left(-\frac{\beta}{x}\right)\right). \tag{36}$$

In above equation N is a constant and γ, β must be $\gamma<-2$, $\beta>0$ [8].

So the spinor wavefunctions of Dirac-Morse problem are also obtained.

4 Conclusion

Here, by using of the shape invariance symmetry in non-relativistic quantum mechanics, we have obtained the spectra of the bound states and spinor wavefunctions of the Dirac equation with spherical symmetry for shifted oscillator and Morse potentials. It is shown that the spinor wavefunctions can be expressed in terms of Rodrigues polynomials of associated differential equations related to each case. It would be interesting to obtain a general formalism for other shape invariant potentials, which is under investigation.

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